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The Dirac equation in Kerr spacetime, spheroidal coordinates and the MIT bag model of hadrons

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Abstract. The Dirac equation in Kerr spacetime is separated using the rotating tetrad formalism. This allows solutions of the Dirac equation, in flat spacetime, written in oblate and prolate spheroidal coordinates to be extracted. The usual MIT bag boundary condition, $-\mathrm{i}\gamma^{\mu}n_{\nu}\Psi = \Psi$ is then found to be incompatible with a non-vanishing separated wavefunction except in the spherical limit. However, it is shown that an alternative boundary condition exists that is physically motivated and allows for a non-trivial solution.

1. Introduction

The object of this paper is twofold. First the Dirac equation written in the rotating tetrad formalism is separated in a Kerr metric background. Second the solutions of the Dirac equation in oblate and prolate spheroidal coordinates are extracted and the possibility of implementing bag-type boundary conditions is investigated.

2. The Dirac equation in the Kerr metric

Schrödinger (1938) was the first to examine the Weyl field in the spacetime of the Schwarzchild metric. At a later time Brill and Wheeler (1957) continued this investigation. Subsequently Teukolsky (1973) and Unruh (1973) used the Newman–Penrose spinor technique to separate the Weyl field in Kerr spacetime. However, it was left to Chandrasekhar (1976, 1983) to separate the full Dirac equation in Kerr spacetime, again using the Newman–Penrose formalism (see also Kalnis and Miller 1991).

Unruh chose to present his results for the Weyl equation using the more familiar rotating tetrad formalism. We have reworked and expanded Unruh's presentation to separate the Dirac equation in Kerr spacetime. Our calculation differs from Unruh's in that the fermion mass is non-zero. We also display the spin connections for this representation explicitly. We hope that our paper will be accessible to readers not familiar with the Newman–Penrose formalism.

The Kerr metric in Boyer–Lindquist coordinates is

$$\mathrm{d}s^2 = -\frac{\Sigma}{\Delta}\mathrm{d}r^2 - \Sigma\,\mathrm{d}\theta^2 + \frac{\Delta}{\Sigma}(\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\phi)^2 - \frac{\sin^2\theta}{\Sigma}(-a\,\mathrm{d}t + (r^2+a^2)\mathrm{d}\phi)^2 \qquad (1)$$

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where $\Delta = r^2 + a^2 - 2Mr$ and $\Sigma = r^2 + a^2\cos^2\theta$. In curved spacetime the Dirac equation takes the form

$$(\gamma^\mu(\partial_\mu - \Gamma_\mu) + \mathrm{i}m_e)\Psi = 0 \tag{2}$$

where the γ^μ and spin connections Γ_μ satisfy

$$[\gamma^\mu, \gamma^\nu]_+ = 2\mathbf{I}g^{\mu\nu} \tag{3}$$

and

$$[\Gamma_\mu, \gamma^\nu]_- = \partial_\mu\gamma^\nu + \Gamma^\nu_{\alpha\mu}\gamma^\alpha \tag{4}$$

where $\Gamma^\nu_{\alpha\mu}$ is the Christoffel symbol corresponding to the particular metric used and $\mathbf{I}$ is the unit matrix in spinor space.

An appropriate choice of the γ^μ and Γ_μ which satisfy (3) and (4) in Kerr spacetime is

$$\begin{aligned} \gamma^t &= \frac{(r^2+a^2)}{(\Sigma\Delta)^{1/2}}\gamma^0 + \frac{a\sin\theta}{\Sigma^{1/2}}\gamma^2 \qquad \gamma^r = \left(\frac{\Delta}{\Sigma}\right)^{1/2}\gamma^3 \\ \gamma^\phi &= \frac{a}{(\Sigma\Delta)^{1/2}}\gamma^0 + \frac{1}{\Sigma^{1/2}\sin\theta}\gamma^2 \qquad \gamma^\theta = \left(\frac{1}{\Sigma}\right)^{1/2}\gamma^1 \end{aligned} \tag{5}$$

and

$$\begin{aligned} \Gamma_r &= \frac{1}{2}\frac{1}{\Sigma}\frac{1}{\Delta^{1/2}}(\gamma^1\gamma^3 a^2\cos\theta\sin\theta + \gamma^0\gamma^2 ar\sin\theta) \\ \Gamma_t &= \frac{1}{2}\frac{M}{\Sigma^2}(\gamma^0\gamma^3(a^2\cos^2\theta - r^2) + \gamma^1\gamma^2 2ar\cos\theta) \\ \Gamma_\theta &= \frac{1}{2}\frac{\Delta^{1/2}}{\Sigma}(\gamma^1\gamma^3 r - \gamma^0\gamma^2 a\cos\theta) \\ \Gamma_\phi &= \frac{1}{2}\frac{1}{\Sigma}\left(\gamma^0\gamma^3 a\sin^2\theta\left(\frac{M}{\Sigma}(r^2 - a^2\cos^2\theta) + r\right) - \gamma^1\gamma^2\cos\theta\right. \\ &\quad \left.\times\left((r^2+a^2) + \frac{2Mr}{\Sigma}a^2\sin^2\theta\right) + \Delta^{1/2}\sin\theta(\gamma^0\gamma^1 a\cos\theta + \gamma^2\gamma^3 r)\right) \end{aligned} \tag{6}$$

where $\gamma^i (i = 0, 1, 2, 3)$ are the usual Bjorken–Drell (Bjorken and Drell 1964) gamma matrices.

In the limit $a \to 0$ the gamma matrices (5) and spin connections (6) reduce to those of Brill and Wheeler (1957) when the following substitutions are made; $\gamma^3 \to -\mathrm{i}\hat{\gamma}^1$, $\gamma^1 \to -\mathrm{i}\hat{\gamma}^2$, $\gamma^2 \to -\mathrm{i}\hat{\gamma}^3$, and $\gamma^0 \to \mathrm{i}\hat{\gamma}^0$, where $\hat{\gamma}^\mu$ are Brill and Wheeler gamma matrices. The factor of $-\mathrm{i}$ comes from the opposite sign of the metric chosen by the authors. These substitutions correspond to a similarity transformation connecting the representations of the Clifford algebra.

Substituting (5) and (6) into (2) yields an equation which is separable if the wavefunction is assumed to take the form

$$\Psi = (\Delta\sin^2\theta)^{-1/4}\mathrm{e}^{+\mathrm{i}(m\phi+\omega t)}\left(\rho^{*-1/2}\begin{pmatrix}\eta^+\\ \eta^+\end{pmatrix} + \rho^{-1/2}\begin{pmatrix}\eta^-\\ -\eta^-\end{pmatrix}\right) \tag{7}$$

where $\rho = (r + \mathrm{i}a\cos\theta)$ and η^+, η^- are as yet to be determined functions of r and θ. (The similarity transformation connecting this solution with that of Chandrasekhar (1976, 1983) is given in the appendix.) The resulting set of equations can be written as

$$\Delta^{-1/2}\left(\pm\mathrm{i}\left(\omega\frac{r^2+a^2}{\Delta}+\frac{am}{\Delta}\right)-\sigma_3\partial_r\right)\eta^{\mp}\pm\mathrm{i}m_e r\eta^{\pm} = \left(\sigma_1\partial_\theta+\mathrm{i}\sigma_2\left(a\omega\sin\theta+\frac{m}{\sin\theta}\right)\right)\eta^{\mp}+am_e\cos\theta\eta^{\pm} \tag{8}$$

where the $\sigma_i (i = 1, 2, 3)$ are the Pauli matrices. It is convenient to consider the separation of the equations in the $m_e \neq 0$ and $m_e = 0$ cases individually.

When $m_e \neq 0$ the two chirality eigenstates are coupled. If one assumes that

$$\eta^+ = \begin{pmatrix} R_1(r) & S_1(\theta) \\ R_2(r) & S_2(\theta) \end{pmatrix} \qquad \eta^- = \begin{pmatrix} R_2(r) & S_1(\theta) \\ R_1(r) & S_2(\theta) \end{pmatrix} \tag{9}$$

then the following two sets of coupled ordinary differential equations result

$$\Delta^{1/2}\left(+\mathrm{i}\left(\omega\frac{r^2+a^2}{\Delta}+\frac{am}{\Delta}\right)-\sigma_3\partial_r\right)\begin{pmatrix} R_2 \\ R_1 \end{pmatrix}+\mathrm{i}m_e r\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = k\begin{pmatrix} R_1 \\ -R_2 \end{pmatrix} \tag{10}$$

$$\left(\sigma_1\partial_\theta+\mathrm{i}\sigma_2\left(a\omega\sin\theta+\frac{m}{\sin\theta}\right)-am_e\cos\theta\right)\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = k\begin{pmatrix} S_1 \\ -S_2 \end{pmatrix} \tag{11}$$

where the separation spinor is $(k, -k)^T$. This explicitly demonstrates that the Dirac equation in a Kerr background separates in the representation of the gamma matrices given by (5).

If $m_e = 0$ the equations for η^+ and η^- decouple and can be considered independently. Assuming that the individual elements of the spinor separate, so that

$$\eta^{\pm} = \begin{pmatrix} R_1^{\pm}(r) & S_1^{\pm}(\theta) \\ R_2^{\pm}(r) & S_2^{\pm}(\theta) \end{pmatrix} \tag{12}$$

the $m_e = 0$ form of (8) then separates into two sets of coupled equations for the r and θ dependent functions:

$$\Delta^{1/2}\left(\pm\mathrm{i}\left(\omega\frac{r^2+a^2}{\Delta}+\frac{am}{\Delta}\right)-\sigma_3\partial_r\right)\begin{pmatrix} R_1^{\mp} \\ R_2^{\mp} \end{pmatrix} = \begin{pmatrix} k_1^{\mp}R_2^{\mp} \\ -k_2^{\mp}R_1^{\mp} \end{pmatrix} \tag{13}$$

$$\left(\sigma_1\partial_\theta+\mathrm{i}\sigma_2\left(a\omega\sin\theta+\frac{m}{\sin\theta}\right)\right)\begin{pmatrix} S_1^{\pm} \\ S_2^{\pm} \end{pmatrix} = \begin{pmatrix} k_1^{\pm}S_1^{\pm} \\ -k_2^{\pm}S_2^{\pm} \end{pmatrix}. \tag{14}$$

The constants $k_1^{\pm}$ and $-k_2^{\pm}$ are the upper and lower components of the two separation two-spinors. However, if the two-spinors $\eta^{\pm}$ are multiplied by the constant $(k_2^{\pm}/k_1^{\pm})$ (which can be absorbed into the normalization) then (13) and (14) take the same form as the massless limit of (10) and (11) with either $k^2 = k_1^+k_2^+ = (k^+)^2$ or $k^2 = k_1^-k_2^- = (k^-)^2$ as appropriate. Thus if $m_e = 0$ the solutions separate into the chiral eigenstate components of the massive solution with $m_e = 0$ and independent separation constants k^+, k^-.

In terms of the arbitrary constants C_1 and C_2, the solutions of the angular equation (11) are

$$S_1 = C_1\sin^{1/2}\theta\,{}_{-1/2}S_{lm}(\theta,\phi)\mathrm{e}^{-\mathrm{i}m\phi} \tag{15}$$

$$S_2 = C_2\sin^{1/2}\theta\,{}_{+1/2}S_{lm}(\theta,\phi)\mathrm{e}^{-\mathrm{i}m\phi} \tag{16}$$

where $k^2 = (l+\frac{1}{2})^2$ and ${}_{\pm 1/2}S_{lm}(\theta,\phi)$ are the mass-dependent spheroidal harmonics of spin one half (Chakrabarti 1984) which can be expressed as a sum of spin-weighted spherical harmonics.

3. Oblate and prolate spheroidal coordinates

It has already been pointed out that the Dirac equation is separable in oblate (Chandrasekhar 1976, 1983, Cook 1982, Kalnis and Miller 1991) and prolate (Cook 1982) spheroidal coordinates (henceforth refered to as OSC and PSC respectively). The most direct way to extract the solution in OSC is to take the appropriate limit of the solution in Kerr spacetime. It has not been mentioned before that a further simple transformation then yields the solution in PSC. We therefore present the two solutions below.

The spatial metric in OSC (η, θ, ϕ) is

$$g_{\eta\eta} = g_{\theta\theta} = a^2(\cosh^2\eta - \sin^2\theta) \qquad g_{\phi\phi} = a^2(\cosh\eta\sin\theta)^2 \tag{17}$$

while the spatial metric in PSC (η', θ', ϕ') is

$$g_{\eta'\eta'} = g_{\theta'\theta'} = a^2(\sinh^2\eta' + \sin^2\theta') \qquad g_{\phi'\phi'} = a^2(\sinh\eta'\sin\theta')^2. \tag{18}$$

where $0 < \eta, \eta' < \infty$, $0 < \theta, \theta' < \pi$ and $0 < \phi, \phi' < 2\pi$.

If the substitutions, $r = a\sinh\eta$ and $r' = a\cosh\eta'$ are made in (17) and (18) respectively then the metrics become

$$g_{\eta\eta} = -\frac{\Sigma^+}{\Delta^+} \qquad g_{\theta\theta} = -\Sigma^+ \qquad g_{\phi\phi} = -\Delta^+\sin^2\theta \tag{19}$$

and

$$g_{\eta'\eta'} = -\frac{\Sigma^-}{\Delta^-} \qquad g_{\theta'\theta'} = -\Sigma^- \qquad g_{\phi'\phi'} = -\Delta^-\sin^2\theta \tag{20}$$

where $\Delta^\pm = (r^2 \pm a^2)$ and $\Sigma^\pm = (r^2 \pm a^2\cos^2\theta)$, while $0 \leqslant r < \infty$ and $a \leqslant r' < \infty$. If alternatively the substitutions $r = -a\sinh\eta$ and $r' = -a\cosh\eta$ are made, then the metric transforms in the same way but the domains of r and r' becomes negative. In the limit $a \to 0$ one expects to find the metric in the spherical polar coordinates, thus the transformation without the minus signs is the case of physical interest. Comparison with (1) reveals that the Kerr metric reduces to the OSC metric in the limit $(M \to 0)$. It follows then, that by taking the limit $M \to 0$, all the equations of the previous section will apply to the Dirac equation in OSC.

Given the expressions describing the Dirac equation in OSC one can recover the appropriate results in PSC by the further transformation $a \to \pm ia$ since this substitution transforms the metrics (19) and (20) into one another. It is also important to redefine the domain of r appropriately. Thus the plus and minus signs in the transformation give rise to two different representations of the gamma matrices (an analogous pair of representations exist in OSC). For simplicity we shall henceforth only consider the transformation involving the positive sign.

There exists an interesting symmetry between the radial and angular equations for the Dirac equation in both OSC and PSC. If the substitution $a\cos\theta = ir$ such that $\sin\theta = \pm\frac{1}{2}(r^2 + a^2)^{1/2}$ is made in the angular equations in OSC then the resulting system of equations is identical to the radial equation in OSC when the further substitutions $((S_2 = R_2, iS_1 = R_1)$ or $(-iS_2 = R_2, S_1 = R_1))$ and $((S_2 = -R_2, iS_1 = R_1)$ or $(-iS_2 = R_2, S_1 = -R_1))$ are made corresponding to the plus and minus square root respectively. Similarly if the substitution $a\cos\theta = r$ such that $\sin\theta = \pm\frac{1}{2}i(r^2 - a^2)^{1/2}$ is made in the angular equations in PSC then the resulting system of equations is identical to the radial equations in PSC when the aforementioned substitutions are made. A similar symmetry exists between the radial and angular parts of the spheroidal wavefunction (see Abramowitz and Stegun 1964).

4. The oblate and prolate spheroidal bags

The MIT bag model of quark confinement is often employed in a variety of different calculations. In most cases the bag boundary is assumed to be a static sphere, it would be useful to extend this model by considering non-spherical boundaries. A spheroidal boundary would be a convenient choice because it can be parametrized by two constants 'a' and 'r_0' which constitute, respectively, a measure of the deviation from spherical symmetry and a generalized radius.

It is then natural to attempt to implement the bag boundary condition on the solutions of the Dirac equation in OSC and PSC. The quark field in the bag model satisfies the boundary condition

$$-\mathrm{i}\gamma^{\mu}n_{\mu}\Psi = \Psi \tag{21}$$

where n_μ is the unit vector in the outward normal direction. For the spheroidal surface $r = r_0$

$$n_\nu = \left(0, \left(\frac{\Sigma^\pm}{\Delta^\pm}\right)^{1/2}, 0, 0\right) \qquad \text{and} \qquad \gamma^r = \left(\frac{\Delta^\pm}{\Sigma^\pm}\right)^{1/2}\gamma^3. \tag{22}$$

Substitution of (7), (12) and (22) into (21) yields the following two equations (applicable to massive fermions)

$$\mathrm{i}R_2(r_0) = \left(\frac{\rho^{*\pm}}{\rho^\pm}\right)^{-1/2} R_1(r_0) \qquad \mathrm{i}R_2(r_0) = \left(\frac{\rho^{\pm}}{\rho^{*\pm}}\right)^{-1/2} R_1(r_0). \tag{23}$$

The system of equations (23) has the unique solution, $R_1(r_0) = R_2(r_0) = 0$, but this is a fixed point of the system of differential equations (13) and thus the wavefunction must vanish everywhere. It is only in the spherical limit that the equations are consistent and that a non-trivial separable solution can be sought.

In the limit $a \to 0$ the pairs of differential equations (10) and (11) can be solved, yielding the solution of the Dirac equation with a rotating tetrad in spherical polar coordinates. The radial solutions are

$$R_1(r) = A_0\frac{r}{2}\left(j_k(\kappa r) + \mathrm{i}\left(\frac{\omega + m_e}{\omega - m_e}\right)^{1/2} j_{k-1}(\kappa r)\right) \tag{24}$$

and

$$R_2(r) = R_1^*(r) \tag{25}$$

where A_0 is an arbitrary constant and $\kappa^2 = \omega^2 - m_e^2$. Substitution of (24) and (25) into (23) with $\rho^{\pm *} = \rho^\pm$ yields the usual expression derived from the solution of the Dirac equation with a fixed tetrad in polar coordinates

$$j_k(\kappa r_0) = \left(\frac{\omega + m_e}{\omega - m_e}\right)^{1/2} j_{k-1}(\kappa r_0).$$

In the massless limit the two helicity eigenstates can be considered separately, the resulting systems of coupled equations again require the wavefunction to vanish. It is only in the spherical limit that two helicity eigenstates (with the same separation constants) can be combined together to give a non-trivial solution which satisfies the boundary condition.

It follows that these separated solutions to Dirac's equation cannot be confined within a static spheroidal MIT bag. This somewhat surprising result has been noted elsewhere (Kalnis and Miller 1991) and can be readily confirmed via Chandrasekhar's solution. However, this result does not obviously exclude the possibility that a linear combination of solutions exists, such that the boundary condition is implementable (see for example Hahn *et al* 1983). Instead of pursuing this alternative, we investigate the possibility that a bag-like solution can be constructed using some generalization of the usual MIT condition (21).

From a strictly mathematical point of view, it is possible to show that more general types of boundary conditions than (21) can be consistently introduced. The requirement that the adjoint Dirac operator be properly defined as a differential operator implies that the boundary condition must take the form

$$\pm\, \mathrm{i}\gamma^{\mu} n_{\mu} X \Psi = \Psi \tag{27}$$

where X satisfies $X^{\dagger}X = 1$ and $\gamma^{\mu}n_{\mu}X = X^{\dagger}\gamma^{\mu}n_{\mu}$ (Luckock 1991). For an arbitrary boundary the most general form possible is

$$-\,\mathrm{i}\mathrm{e}^{-\mathrm{i}\beta\gamma^{5}}\gamma^{\mu}n_{\mu}\Psi = \Psi \tag{28}$$

where β is some arbitrary function of the coordinates on the boundary. (We have chosen the sign so as to agree with the parity of the usual MIT condition.) Interestingly, if we choose the function β to be

$$\beta = \tan^{-1}(a\cos\theta/r) \tag{29}$$

then the following sensible eigenvalue problem results from substituting the separated solution (7) into (27)

$$R_1(r_0) = \mathrm{i}R_2(r_0). \tag{30}$$

In effect, the chiral phase β cancels the unwanted θ-dependent factors in (23) since $\rho = |\rho|\mathrm{e}^{\mathrm{i}\beta}$. Thus, it is possible to impose consistent bag-like boundary condition (28) and (29), on the separated solution (7), over a spheroidal surface.

What then, is the physical content of (28) and (29)? The usual MIT boundary condition (21) is sufficient to ensure that both the outward current $\mathrm{i}\bar{\Psi}n_{\mu}\gamma^{\mu}\Psi$ and the density $\bar{\Psi}\Psi$ vanish on the boundary. In the case of the generalized condition (28), the current vanishes but $\bar{\Psi}\Psi$ does not. In its place, the following expression is fixed to zero on the surface

$$\bar{\Psi}(\cos\beta + \mathrm{i}\gamma^{5}\sin\beta)\Psi. \tag{31}$$

From the definition of the current, the form of the solution (7), and the equation of motion (10), it is easy to show that the current in the r direction always vanishes within the bag. Thus (28) and (29) provide a sensible boundary condition for modelling hadrons since they ensure that the current is continuous at the boundary. Recall that it is the current, rather than $\bar{\Psi}\Psi$, that carries unambiguous physical significance. Having also noticed that the MIT boundary condition (21) cannot be implemented with separated solutions in OSC, Kalnis and Miller (1991) have proposed the alternative condition

$$-\,\mathrm{i}\mathrm{e}^{-\mathrm{i}\beta\gamma^{0}}\gamma^{\mu}n_{\mu}\Psi = \Psi. \tag{32}$$

This suffers from two drawbacks. First, in general it fails to satisfy Luckock's condition $\gamma^\mu n_\mu X = X^\dagger \gamma^\mu n_\mu$ and becomes inconsistent for an arbitrarily shaped bag boundary. Second, even if the boundary is constrained so that (32) is self-consistent, the current through the boundary becomes non-zero although the density $\bar{\Psi}\Psi$ does vanish. Since this introduces a discontinuity in the current, it is tempting to speculate that (32) implies the presense of a scalar plus electrostatic potential at the boundary—the current at the boundary might then be explained via the Klien paradox due to the electrostatic component of the potential. In any case we shall not consider this possibility further.

It is possible to provide a physical motivation for (28) along lines similar to those argued by Chodos *et al* (1974) in their original development of the MIT model. Consider a combination of scalar and pseudoscalar potentials that can be written as

$$V_1 + \mathrm{i}\gamma^5 V_2 = V\mathrm{e}^{\mathrm{i}\beta\gamma^5} \tag{33}$$

in an obvious notation. The Dirac equation in this potential is then

$$(\mathrm{i}\gamma^\mu\partial_\mu - V\mathrm{e}^{\mathrm{i}\beta\gamma^5})\Psi = 0. \tag{34}$$

(For simplicity we have taken $m = 0$ since a non-zero m can always be absorbed into V, but as we are going to take $V \to \infty$ anyway, it will become irrelevant.) Now consider

$$(\mathrm{i}\gamma^\mu\partial_\mu - V)\mathrm{e}^{\mathrm{i}\beta\gamma^5}\Psi = \mathrm{e}^{-\mathrm{i}\beta\gamma^5/2}(\mathrm{i}\gamma^\mu\partial_\mu - V\mathrm{e}^{\mathrm{i}\beta\gamma^5})\Psi + \tfrac{1}{2}\mathrm{e}^{-\mathrm{i}\beta\gamma^5/2}\gamma^\mu\gamma^5(\partial_\mu\beta)\Psi. \tag{35}$$

From the Dirac equation for Ψ the first term on the right vanishes. The remaining term can be taken to be finite as $V \to \infty$, so in the limit of large V, $\mathrm{e}^{\mathrm{i}\beta\gamma^5/2}\Psi$ satisfies the usual Dirac equation in a (large) scalar potential. Following the argument of Chodos *et al* (1974), we can derive the boundary condition

$$\mathrm{e}^{\mathrm{i}\beta\gamma^5/2}\Psi = -\mathrm{i}\gamma^\mu n_\mu \mathrm{e}^{\mathrm{i}\beta\gamma^5/2}\Psi \tag{36}$$

which is equivalent to (28). In the above argument the function β is arbitrary, and no justification for the specific choice (29) has been advanced. At least however, it provides a motivation for considering angle-dependent-type boundary conditions which, given the choice (29), allow a non-trivial eigenvalue problem to be extracted from the separated solutions.

Finally it is interesting to note that the functions $\sin\beta$ and $\cos\beta$ which appear in (31) are related to solutions of Laplace's equation (Birse 1992). Indeed, the general solution of $\nabla^2\sigma = 0$ in spheroidal coordinates is

$$\sigma = C_1\frac{\cos\beta}{\Sigma^{1/2}} + C_2\frac{\sin\beta}{\Sigma^{1/2}}. \tag{37}$$

This tantalizing result may be a hint of the physics that underlies the clearly useful, but only partially explained, boundary conditions (28)–(29).

5. Conclusion

We have presented a separation of Dirac's equation in Kerr spacetime and used the result to study the Dirac equation in oblate and prolate spheroidal coordinates. It was found that the usual MIT bag boundary condition is incompatible with a non-trivial separated solution—except in the spherical limit. A new boundary condition (28) has been proposed which, for an appropriate choice of β, results in a sensible eigenvalue problem. Furthermore, it has been shown that a chirally rotated boundary condition of the form (28) can arise naturally in scalar–pseudoscalar potential models. Thus we have demonstrated that the new boundary conditions (28)–(29) have a practical and, to some extent physical, motivation. Clearly though, more work will be required before this curious result is fully understood.

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Appendix

In this appendix we relate Chandrasekhar's (1976, 1983) solution of the Dirac equation in a Kerr background to the one developed here. We shall adopt the notation and equation enumeration of Chandrasekhar (1983) (chapter 10).

Chandrasekhar's equation (109) can be rewritten in the usual Dirac equation form only if the mass is rescaled so that

$$\mu_* = 2^{-1/2} m_e \tag{A.1}$$

Once this is done the gamma matrices in Chandrasekhar's representation, $\gamma^{\mu}_{\mathrm{CH}}$ can be extracted;

$$\begin{aligned}
\gamma^{r}_{\mathrm{CH}} &= \begin{pmatrix} 0 & \sigma_r \\ \sigma_r & 0 \end{pmatrix} \qquad \gamma^{\theta}_{\mathrm{CH}} = \begin{pmatrix} 0 & \sigma_\theta \\ -\sigma_\theta & 0 \end{pmatrix} \\
\gamma^{t}_{\mathrm{CH}} &= \begin{pmatrix} 0 & \sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix} \frac{(r^2+a^2)}{\Delta} + \begin{pmatrix} 0 & \sigma_\beta \\ -\sigma_{\beta *} & 0 \end{pmatrix} a \sin\theta \\
\gamma^{\phi}_{\mathrm{CH}} &= \begin{pmatrix} 0 & \sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix} \frac{a}{\Delta} + \begin{pmatrix} 0 & \sigma_\beta \\ -\sigma_{\beta *} & 0 \end{pmatrix} \frac{1}{\sin\theta}
\end{aligned} \tag{A.2}$$

where

$$\begin{aligned}
\sigma_r &= \begin{pmatrix} 0 & -\Delta/\Sigma \\ 2 & 0 \end{pmatrix} 2^{-1/2} \qquad \sigma_\theta = \begin{pmatrix} -1/\rho * & 0 \\ 0 & -1/\rho \end{pmatrix} \\
\sigma_\alpha &= \begin{pmatrix} 0 & \Delta/\Sigma \\ 2 & 0 \end{pmatrix} 2^{-1/2} \qquad \sigma_\beta = \begin{pmatrix} \mathrm{i}/\rho * & 0 \\ 0 & -\mathrm{i}/\rho \end{pmatrix}.
\end{aligned}$$

While the corresponding wavefunction takes the form

$$\Psi_{\mathrm{CH}} = (F_1, F_2, -G_2, -G_1)^T. \tag{A.3}$$

These gamma matrices and the wavefunction differ non-trivially from those proposed by Iyer and Kummar (1978). It is straightforward to see that their proposed representation is flawed because their gamma matrices fail to satisfy (3).

It is possible to verify that the two solutions, and the associated representations of the gamma matrices, are related by the similarity transformation

$$S\Psi_{\mathrm{CH}} = \Psi \qquad \text{and} \qquad S\gamma^{\mu}_{\mathrm{CH}} S^{-1} = \gamma^{\mu} \tag{A.4}$$

where

$$S = \Delta^{-1/4} \begin{pmatrix} \sigma_\gamma & \sigma_\delta \\ \sigma_\gamma & -\sigma_\delta \end{pmatrix} \tag{A.5}$$

and

$$\sigma_\gamma = \begin{pmatrix} 2^{1/2}\rho *^{1/2} & 0 \\ 0 & \Delta^{1/2}\rho *^{-1/2} \end{pmatrix} \qquad \sigma_\delta = \begin{pmatrix} 0 & \Delta^{1/2}\rho^{-1/2} \\ 2^{1/2}\rho^{1/2} & 0 \end{pmatrix}. \tag{A.6}$$

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